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Examination of Diffusion Modeling Using Zero-Mean-Shear Turbulence

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An examination of diffusion modeling in second-moment turbulence closures is presented. The main objectives are to gain a better understanding of the processes that are grouped into the net diffusive transport of the Reynolds stresses, $\mathcal{D}_{ij}$, and to assess existing models for $\mathcal{D}_{ij}$. The benchmark case of zero-mean-shear (ZMS) turbulence is used as a guide. The analysis of ZMS turbulence shows that the pressure–velocity processes in $\mathcal{D}_{ij}$ play a central role in establishing the anisotropy level in diffusive turbulence. Existing models for $\mathcal{D}_{ij}$ based solely on the triple velocity correlation are shown to be inadequate in the diffusive limit. The present evaluation of diffusion models using ZMS turbulence and subsequent analysis of ZMS turbulence show that the Lumley (Lumley, J. L., "Computational Modelling of Turbulent Flows," *Advances in Applied Mechanics*, Vol. 18, Academic, New York, 1978, pp. 123–176) model is the most viable existing diffusion model. A modification to Lumley's model that enables exact predictions in the diffusive limit is suggested.

I. Introduction

SECOND-MOMENT or Reynolds stress (RS) turbulence closures are becoming more widely used for the prediction of complex industrial flows.[1,2] Researchers have long been interested in the flexibility and potential universality of RS closures, but only recently, with the widespread availability of powerful computers, have RS closures been used in engineering computations.

The most significant advancements in RS modeling were presented in papers by Rotta,[3] Hanjalić and Launder,[4] and Launder et al.[5]; the latter is referred to as the standard closure. Subsequent advances in RS modeling have, for the most part, been modifications to the standard closure. Significant contributions include, for example, the development of models for low-Reynolds-number turbulence,[6,7] improvement of the pressure–velocity interaction models,[8–11] and models for the source and sink processes in the transport of dissipation rate equation.[9]

Models have also been developed to approximate diffusive transport in the RS equation. A review of prominent existing models is given in the main body of this paper, following a description of the diffusion process. In modern computations involving RS closures, the selection of a diffusion model is usually based more on computational simplicity than on physical integrity.[12–14] The general assertion is that diffusive transport is small in the RS budget and accurate modeling of diffusion is, therefore, nonessential. The modeling community is not convinced that complex models for diffusion are necessary. For the most part, complex diffusion models, some of which involve the solution of third-moment equations, are cumbersome to implement, and they have not been shown to be more physically realistic than their simpler and, in most cases, more computationally stable counterparts. There still exists a lack of information on the turbulent diffusion process, and this, more than anything, contributes to the indiscriminate selection of diffusion models for engineering computations.

In the present study, the modeling of diffusive transport is examined for zero-mean-shear (ZMS) turbulence, a case that represents the diffusive limit. In ZMS turbulence, no mean flow exists, and as such the level of turbulent kinetic energy is maintained by a balance of diffusive transport and dissipation. The anisotropy in ZMS turbulence, which assumes a constant level, is established by a balance of diffusive transport and pressure–velocity interaction. Having these relatively simple characteristics, ZMS turbulence is useful as a benchmark for testing diffusion and pressure–strain models in second-moment closures and for examining the relative influence of the different diffusion processes.

This paper consists of two main parts: a description of diffusive transport in second-moment closures, including a review and evaluation of many existing models, and an examination of ZMS turbulence, including an evaluation of existing second-moment closure models, an analysis of the diffusion process, and a recommendation of the best existing RS diffusion model.

II. Second-Moment Turbulence Modeling

In RS closures, equations for the transport of the individual turbulent stresses are solved along with the conservation equations for mass and momentum. The exact equation for the transport of $\overline{u_i u_j}$ is derived by taking velocity-weighted moments of the momentum equations. The result is expressed in Cartesian tensor form as

$$\underbrace{\frac{\partial \overline{u_i u_j}}{\partial t} + \bar{U}_k \frac{\partial \overline{u_i u_j}}{\partial x_k}}_{\mathcal{C}_{ij}} = \underbrace{\frac{\partial}{\partial x_k}\left(\nu \frac{\partial \overline{u_i u_j}}{\partial x_k} - \overline{u_i u_j u_k} - \frac{\overline{p' u_i}}{\rho}\delta_{kj} - \frac{\overline{p' u_j}}{\rho}\delta_{ki}\right)}_{\mathcal{D}_{ij}} \underbrace{- \left(\overline{u_i u_k}\frac{\partial \bar{U}_j}{\partial x_k} + \overline{u_j u_k}\frac{\partial \bar{U}_i}{\partial x_k}\right)}_{\mathcal{P}_{ij}} + \underbrace{\overline{\frac{p'}{\rho}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)}}_{\pi_{ij}} \underbrace{- 2\nu \overline{\left(\frac{\partial u_j}{\partial x_k}\frac{\partial u_i}{\partial x_k}\right)}}_{\epsilon_{ij}} \quad (1)$$

where $\overline{u_i u_j}$ is the RS tensor, $\bar{U}_k$ is the mean flow vector, p' is the pressure fluctuation, and ρ and ν are the fluid density and kinematic viscosity, respectively. The grouped terms represent the convection

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$\mathcal{C}_{ij}$, diffusion $\mathcal{D}_{ij}$, shear production $\mathcal{P}_{ij}$, pressure–strain π_{ij}, and dissipation ϵ_{ij}, of $\overline{u_i u_j}$. The convection and shear production terms in Eq. (1) require no approximation, but the pressure–strain, dissipation and diffusion of $\overline{u_i u_j}$ must be modeled in terms of dependent variables to obtain a closed, second-moment framework. Experimental and theoretical considerations suggest that the pressure–strain term should be modeled in terms of the anisotropy tensor

$$a_{ij} = \frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij} \tag{2}$$

its invariants

$$A = 1 - \tfrac{9}{8}(A_2 - A_3), \qquad A_2 = a_{ik}a_{ki}, \qquad A_3 = a_{ik}a_{kl}a_{li} \tag{3}$$

the mean strain and rotation

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \bar{U}_i}{\partial x_j} + \frac{\partial \bar{U}_j}{\partial x_i}\right), \qquad W_{ij} = \frac{1}{2}\left(\frac{\partial \bar{U}_i}{\partial x_j} - \frac{\partial \bar{U}_j}{\partial x_i}\right) \tag{4}$$

and the additional tensor

$$D_{ik} = -\left(\overline{u_i u_m}\frac{\partial \bar{U}_m}{\partial x_k} + \overline{u_k u_m}\frac{\partial \bar{U}_m}{\partial x_i}\right) \tag{5}$$

Three different relations are considered in the present work to model the pressure–strain terms. The simplest of these is the relation described by Launder[9] (LNDR),

$$\pi^l_{ij} = -C^l_1 \epsilon a_{ij} - C^l_2\left(\mathcal{P}_{ij} - \tfrac{1}{3}\mathcal{P}_{kk}\delta_{ij}\right) \tag{6}$$

where the expressions on the right-hand side are, respectively, the return-to-isotropy model of Rotta[3] and the isotropization-of-production model of Launder et al.[5] The remaining two pressure–strain models are considerably more complex than Eq. (6), but both have received favorable reviews from the modeling community. They are the models of Fu et al.[8] (FLT),

$$\begin{aligned}
\pi^f_{ij} = {} & -C^f_1 A^{\frac{1}{2}} A_2 \epsilon\left[a_{ij} + C'_1\left(a_{im}a_{mj} - \tfrac{1}{3}A_2\delta_{ij}\right)\right] - a_{ij}\epsilon \\
& - 0.6\left(\mathcal{P}_{ij} - \tfrac{1}{3}\mathcal{P}_{kk}\delta_{ij}\right) + 0.3\, a_{ij}\mathcal{P}_{kk} - 0.2/k(\overline{u_i u_m}\mathcal{P}_{mj} \\
& + \overline{u_m u_j}\mathcal{P}_{im}) - 0.4\overline{u_l u_i}\,\overline{u_k u_j}S_{ik}/k - 0.65[A_2(\mathcal{P}_{ij} - D_{ij}) \\
& + a_{li}a_{kj}(\mathcal{P}_{lk} - D_{lk})]
\end{aligned} \tag{7}$$

and the model of Speziale et al.[10] (SSG),

$$\begin{aligned}
\pi^s_{ij} = {} & -\left(C^s_1\epsilon + C^*_1\mathcal{P}_k\right)a_{ij} + C^s_2\epsilon\left(a_{ik}a_{kj} - \tfrac{1}{3}A_2\delta_{ij}\right) \\
& + \left(C^s_3 - C^*_3 A_2^{\frac{1}{2}}\right)kS_{ij} + C^s_4 k\left(a_{ik}S_{jk} + a_{jk}S_{ik} - \tfrac{2}{3}a_{kl}S_{kl}\delta_{ij}\right) \\
& + C^s_5 k(a_{ik}W_{jk} + a_{jk}W_{ik})
\end{aligned} \tag{8}$$

where $k\ (= \frac{1}{2}\overline{u_i u_i})$ is the turbulent kinetic energy, ϵ is the isotropic dissipation rate of k, and $\mathcal{P}_k = \frac{1}{2}\mathcal{P}_{kk}$ is the production of k.

The dissipation ϵ_{ij} is assumed to be isotropic and is modeled using the conventional $\epsilon_{ij} = \frac{2}{3}\epsilon\delta_{ij}$ model; ϵ is then obtained from the following transport equation:

$$\frac{\partial \epsilon}{\partial t} + \bar{U}_k\frac{\partial \epsilon}{\partial x_k} = \frac{\partial}{\partial x_k}\left[C_\epsilon \frac{k}{\epsilon}\left(\overline{u_k u_l}\frac{\partial \epsilon}{\partial x_l}\right)\right] + \frac{1}{2}C_{\epsilon 1}\frac{\epsilon}{k}\mathcal{P}_{kk} - C_{\epsilon 2}\frac{\epsilon^2}{k} \tag{9}$$

The modeling of $\mathcal{D}_{ij}$ will be considered in the remainder of this paper. Recommended values for the model coefficients appearing in Eqs. (6–9) are

$$\left\{C^l_1, C^l_2\right\} = \{1.8, 0.6\}, \qquad \left\{C^f_1, C'_1\right\} = \{4.0, 1.2\}$$

$$\left\{C^s_1, C^*_1, C^s_2 C^s_3, C^*_3, C^s_4, C^s_5\right\} = \{1.7, 0.9, 1.05, 0.8, 0.65, 0.625, 0.2\}$$

$$\{C_\epsilon, C_{\epsilon 1}, C_{\epsilon 2}\} = \{0.14, 1.44, 1.92\}$$

III. Diffusive Transport of $\overline{u_i u_j}$

Three separate processes are grouped into $\mathcal{D}_{ij}$ [see Eq. (1)] to represent the net diffusive transport of $\overline{u_i u_j}$. The term involving the kinematic viscosity ν represents the diffusion of turbulence moments by the fluid's natural molecular transport processes. The triple velocity correlation represents the transport of turbulence moments by velocity fluctuations, commonly called turbulent transport, and the two terms containing pressure–velocity correlations represent a process that is referred to as pressure–diffusion.

A model must be provided that approximates $\mathcal{D}_{ij}$ in terms of dependent variables, and this requires further consideration of each of its three components. The molecular diffusion component is already in a suitable form, but for high-turbulence Reynolds number Re_t molecular diffusion is small, and as such this term is usually ignored in $\mathcal{D}_{ij}$ models. The turbulent transport component has been studied by consideration of the exact transport equation for $\overline{u_i u_j u_k}$ and by examining direct measurements of triple correlations from various flows. Turbulent transport is considered to be the dominant mechanism in $\mathcal{D}_{ij}$, and past modeling efforts have focused almost exclusively on this process. The pressure–diffusion component is, by comparison, very poorly understood, due mainly to the difficulty in measuring the pressure–velocity correlations in turbulent flows. In most existing models for $\mathcal{D}_{ij}$, pressure–diffusion either has been neglected on the basis that it is small or has been indirectly accounted for by adjusting the coefficient in the turbulent transport model, a process referred to as optimization of $\mathcal{D}_{ij}$. To clarify, when pressure–diffusion is neglected, the coefficient in the turbulence transport model is tuned to give correct approximations for the triple correlations; when optimization of $\mathcal{D}_{ij}$ is used, the coefficient in the turbulent transport model is tuned to give the correct net diffusion in the $\overline{u_i u_j}$ transport equation.

The review presented in the next section describes many of the diffusion models that have been proposed over the past three decades. As part of the review, the proposed models are classified in terms of their implementation as either simple, intermediate, or complex. Following the review of models, a brief section is included to review the existing evaluations of diffusion models. The existing evaluations give a clear indication of which diffusion models are viable, but it also becomes evident that these evaluations are incomplete in terms of examining the entire diffusion process. This leads naturally to the present evaluation of diffusion models using ZMS turbulence and subsequent analysis of ZMS turbulence. In these sections we provide some much needed insight into the diffusion process, insight that allows us to determine the best existing diffusion model.

A. Review of $\mathcal{D}_{ij}$ Models

A simple gradient-transport relation for the triple correlation was proposed by Daly and Harlow[15] (DH):

$$-\overline{u_i u_j u_k} = C^{dh}_s \frac{k}{\epsilon}\left(\overline{u_k u_l}\frac{\partial \overline{u_i u_j}}{\partial x_l}\right) \tag{10}$$

where $C^{dh}_s = 0.22$ is the recommended coefficient based on optimization of $\mathcal{D}_{ij}$. Models similar to Eq. (10) were also proposed by Wyngaard et al.[16] and Shir.[17] The fundamental weaknesses of these models (as models for $\overline{u_i u_j u_k}$) are that they do not exhibit frame invariance and they are symmetric in only i and j, whereas the triple correlation is symmetric in all three indices.

Another relatively simple, single-coefficient relation was proposed by Hanjalić and Launder[4] (HL) and was obtained by consideration of the exact equation for the transport of $\overline{u_i u_j u_k}$. HL simplified the transport equation for $\overline{u_i u_j u_k}$ to the extent that the following algebraic expression emerged:

$$-\overline{u_i u_j u_k} = C^{hl}_s (k/\epsilon)\mathcal{G}_{ijk} \tag{11}$$

where

$$\mathcal{G}_{ijk} = \left(\overline{u_i u_l}\frac{\partial \overline{u_j u_k}}{\partial x_l} + \overline{u_j u_l}\frac{\partial \overline{u_i u_k}}{\partial x_l} + \overline{u_k u_l}\frac{\partial \overline{u_i u_j}}{\partial x_l}\right) \tag{12}$$

$C^{hl}_s = 0.11$ is the modern recommended coefficient, again based on optimization of $\mathcal{D}_{ij}$. The HL model represents a gradient-transport approximation that is frame invariant and satisfies the full symmetry of the triple correlation. A simplified version of the HL model was proposed by Mellor and Herring[18] (MH) whereby the turbulent diffusivities were assumed to be isotropic instead of tensorial. The MH model is given as

$$-\overline{u_i u_j u_k} = C_s^{mh} \frac{k^2}{\epsilon} \left(\frac{\partial \overline{u_j u_k}}{\partial x_i} + \frac{\partial \overline{u_i u_k}}{\partial x_j} + \frac{\partial \overline{u_i u_j}}{\partial x_k} \right) \tag{13}$$

where C_s^{mh} has a value of 0.20, again based on optimization of $\mathcal{D}_{ij}$.

Another model, which is classed here as a simple $\mathcal{D}_{ij}$ model, is the relation proposed by Lumley[19] (LUM), where separate models were devised for the triple velocity correlation and pressure–diffusion components of $\mathcal{D}_{ij}$. The model for $\overline{u_i u_j u_k}$ was derived from statistical considerations in weakly inhomogeneous flows and is given as

$$-\overline{u_i u_j u_k} = C_{s1}^l (k/\epsilon)\left[\mathcal{G}_{ijk} + C_{s2}^l(\mathcal{G}_i \delta_{jk} + \mathcal{G}_j \delta_{ik} + \mathcal{G}_k \delta_{ij})\right] \tag{14}$$

where

$$\mathcal{G}_i = \mathcal{G}_{imm} \tag{15}$$

The terms in parentheses, multiplied by C_{s2}^l, essentially provide a link between the $\mathcal{G}_{ijk}$ correlations. The LUM model for the pressure–diffusion component of $\mathcal{D}_{ij}$ is given as

$$(1/\rho)\overline{p' u_k} = -\tfrac{1}{5}\overline{u_k u_m u_m} \tag{16}$$

$C_{s1}^l = 1/3C_1 = 0.098$ and $C_{s2}^l = (C_1 - 1)/(4C_1 + 5) = 0.129$ (for $C_1 = 3.4$) are the recommended coefficients for compatibility with the SSG pressure–strain relation.[20]

Models of intermediate complexity but still based on the gradient–transport approximation were proposed by Cormack et al.[21] and Magnaudet.[22] Both of these models were derived from a general asymptotic expansion of $\overline{u_i u_j u_k}$ about an isotropic, homogeneous state using the method of invariant modeling originally proposed by Lumley and Khajeh-Nouri.[23] Like the HL model, the resulting relations are frame invariant and symmetric in all three indices. The Cormack et al.[21] model contains 10 gradient-transport tensor terms and 4 independent coefficients, which were set to agree with triple correlation measurements. The Magnaudet[22] model for $\overline{u_i u_j u_k}$ is similar to the Cormack et al.[21] model, but in the model for $\mathcal{D}_{ij}$, the LUM model for pressure–diffusion is also included.

The most complicated models for $\overline{u_i u_j u_k}$ are not based on the gradient-transport approximation but instead are based on approximate solutions of the exact transport equation for $\overline{u_i u_j u_k}$. These include the models of Amano et al.[24] and Nagano and Tagawa.[25] In both models, coefficients in the approximate transport equation for $\overline{u_i u_j u_k}$ were set to obtain solutions that agreed with triple correlation measurements.

B. Previous Evaluations of $\mathcal{D}_{ij}$ Models

Evaluation of the mentioned models has mainly been by direct comparisons with triple correlation measurements. Cormack et al.[21] included detailed comparisons between the results of existing gradient-transport models and triple correlation data from asymmetric channel flow, wall jets, turbulent pipe flow, and mixing layers. In these comparisons, the models of DH and Shir[17] gave poor agreement with the data and their own model, and that of HL offered good overall agreement.

Amano et al.[24] included comparisons of triple correlations predicted by their transport model and several gradient-transport models (DH, Shir,[17] HL, and Cormack et al.[21]) for flow over a backward-facing step. Although the Amano et al.[24] model was slightly better in some regions, in general, the comparisons favored the HL model as being the most reliable overall.

Model comparisons were also included in Ref. 25. Predictions of triple correlations for 12 different shear flows suggest that this model may be preferable to all existing models for the triple correlation. The main deterrent against using the Nagano and Tagawa[25] model in general applications is its computational complexity.

A further evaluation of gradient-transport models (DH, HL, MH, and LUM) was included by Schwarz and Bradshaw.[20] Their evaluation was based on comparisons between model predictions and triple correlation data from a three-dimensional boundary layer. The predictions from the LUM and HL models, respectively, were in best overall agreement with the data. The predictions from the DH model were consistently farthest from the data, and the predictions from the MH model fell between those of the DH and HL models. Schwarz and Bradshaw[20] concluded that the assumption of an isotropic turbulent diffusivity, as in the DH and MH models, leads to poor predictions in three-dimensional flows.

In summary, the existing evaluations clearly favor the simple gradient-transport relations for modeling diffusion. Although there are weaknesses in these simple models, made evident by the fact that predictions for certain flow quantities are poor, the preceding evaluations provide no justification to jettison the gradient-transport approach in favor of the more complex approaches. It is also important to note that it is impossible to determine the exact nature of the weaknesses in diffusion models on the basis of the shear-flow data used in previous evaluations. In shear flows, even those where diffusion is significant, the intricacies of diffusive transport are obfuscated by the comensurate contributions of other processes. Thus, only the net diffusive transport can be determined with any accuracy. To determine the effects and the relative contributions of the different diffusion processes, and thus the model weaknesses, a more diffusive flow should be used for evaluation. In the next section, the simple gradient-transport diffusion models are evaluated using ZMS turbulence. This evaluation provides insight into the main weaknesses of $\mathcal{D}_{ij}$ models, and a subsequent analysis of ZMS turbulence provides further insight to overcome these weaknesses.

IV. ZMS Turbulence

ZMS turbulence is obtained by allowing turbulence generated by a planar source to diffuse into a fluid in the absence of mean flow. Because there is no mean flow, the transport of turbulent kinetic energy k from the source layer into and throughout the fluid is entirely diffusive. Further, there is no production of turbulence except in the source layer. This means that the spatial variation of k is established by a balance of diffusive transport and dissipation. An important characteristic of ZMS turbulence, which is evidently a result of this balance, is that a spatially constant level of anisotropy is maintained between the planar and transport directions. The anisotropy in ZMS turbulence is generated in the diffusion process and maintained by a balance between diffusion and pressure–strain.

It is relatively straightforward to derive expressions for the spatial variation of turbulence and the growth rate of integral length scales in steady, high-turbulence-Reynolds-number ZMS turbulence (see, for example, Ref. 26). The resulting expressions are given as

$$\bar{u}_1 \propto x_3^{-n}, \qquad l = \beta x_3, \qquad a = (\overline{u_3 u_3}/\overline{u_1 u_1})^{\frac{1}{2}} \approx \text{const} \tag{17}$$

where n, β, and a have been established experimentally to be $n = 1.0$, $\beta = 0.1 - 0.25$, and $a \approx 1.2$ [see Table 1 (Refs. 26–33)]. The Appendix contains further discussion of the inverse power law for $\bar{u}_1$ and a verification of the high-turbulence-Reynolds-number assumption that forms the basis of the inverse power law for $\bar{u}_1$.

Table 1 Summary of values determined for ZMS turbulence parameters in various experimental studies

Reference	n	β	a[a]
Thompson and Turner[27]	1.5	0.1	——
Hopfinger and Toly[26]	1.0	0.1–0.25	1.2
E and Hopfinger[28]	1.0	0.1–0.25	——
Nokes[29]	0.8–1.2	0.1	——
Hannoun et al.[30]	1.0	0.1	1.32
De Silva and Fernando[31]	1.0	0.1	1.18
De Silva and Fernando[32]	1.0	0.1	1.18
Kit et al.[33]	1.0	0.125	1.1

[a]Not all studies provided measurements for a.

A. Evaluation of $\mathcal{D}_{ij}$ Models Using ZMS Turbulence

Attention is now turned to the evaluation of computational models using ZMS turbulence. To take advantage of the simple spatial structure of ZMS turbulence, an approach has been devised whereby model predictions can be made analytically, thus eliminating any inference of numerical error in the evaluations. We begin by considering the desired results, i.e., the prediction of n and a for the expressions in Eq. (17). (The prediction of β is determined by solving the dissipation rate equation and, as will be seen subsequently, has no influence on the other predictions.) Concerning the prediction of n, we note that, in general, the exponent in a power-law expression is extremely sensitive to the position of the origin. In experimental studies of ZMS turbulence (see, for example, Ref. 26), the concept of the virtual origin is used to quantify measured results. The virtual origin is determined by extrapolating the integral length scale variation [see Eq. (17)] to zero and represents the origin for the $\bar{u}_1$ expression. In terms of our analytical predictions, the position of the origin (virtual origin), and thus the value of n, is controlled by the coefficients and the imposition of boundary conditions in the dissipation rate equation. Because our main interest here is to evaluate the performance of models in the RS equation ($\mathcal{D}_{ij}$ models in particular), we assume in the present analysis that the origin is situated to give the inverse power law for $\bar{u}_1$. Thus, the result that this evaluation focuses on is the anisotropy level a. The present approach is devised to give a prediction of a for each combination of diffusion and pressure–strain models for the case where the power-law exponent n has a value of 1. In this respect, the present evaluation focuses exclusively on the result that the models of interest directly influence.

To begin, the simplified RS equations for steady ZMS turbulence diffusing from the $x_1 - x_2$ plane in the x_3 direction are given as

$$0 = \mathcal{D}_{11} + \pi_{11} - \tfrac{2}{3}\epsilon \tag{18}$$

$$0 = \mathcal{D}_{33} + \pi_{33} - \tfrac{2}{3}\epsilon \tag{19}$$

Because all of the gradient-transport diffusion models to be evaluated can be rewritten as the product of a coefficient and a tensor (minor algebraic manipulation is required for the LUM model), Eqs. (18) and (19) can be rewritten as

$$0 = C_s\mathcal{D}^c_{11} + \pi_{11} - \tfrac{2}{3}\epsilon \tag{20}$$

$$0 = C_s\mathcal{D}^c_{33} + \pi_{33} - \tfrac{2}{3}\epsilon \tag{21}$$

where the superscript c indicates that $\mathcal{D}$ is the diffusion component without the coefficient and C_s is the coefficient for the particular diffusion model used. Equations (20) and (21) can each be solved for C_s and then equated to give

$$\frac{\mathcal{D}^c_{33}}{\mathcal{D}^c_{11}} = \frac{\frac{2}{3} - \pi_{33}/\epsilon}{\frac{2}{3} - \pi_{11}/\epsilon} \tag{22}$$

which, when modeled, can be used to obtain a value of a for each combination of pressure–strain and diffusion models. The three different pressure–strain relations, LNDR, FLT, and SSG, valid for ZMS turbulence are given as

$$\pi^l_{ij} = -C^l_1\epsilon a_{ij} \tag{23}$$

$$\pi^f_{ij} = -C^f_1 A^{\frac{1}{2}} A_2\epsilon\left[a_{ij} + C'_1\left(a_{im}a_{mj} - \tfrac{1}{3}A_2\delta_{ij}\right)\right] - a_{ij}\epsilon \tag{24}$$

$$\pi^s_{ij} = -C^s_1\epsilon a_{ij} + C^s_2\epsilon\left(a_{ik}a_{kj} - \tfrac{1}{3}A_2\delta_{ij}\right) \tag{25}$$

where the coefficients are given in Sec. II. The four gradient-transport diffusion models considered in this evaluation, DH, MH, HL, and LUM, are given in Eqs. (10), (11), (13), and (14), respectively.

Next, using the relations in Eq. (17) and the value $n = 1$, the following solution fields are devised for ZMS turbulence:

$$\overline{u_1u_1} = \overline{u_2u_2} = B^2/x_3^2, \qquad \overline{u_3u_3} = a^2\left(B^2/x_3^2\right)$$
$$k = \tfrac{1}{2}\overline{u_iu_i} = [(2+a^2)/2]\left(B^2/x_3^2\right) \tag{26}$$

where B is an arbitrary coefficient that is dependent on the strength of the turbulence source.

Table 2 Components of the pressure–strain models valid for ZMS turbulence

Model	π_{11}/ϵ	π_{33}/ϵ
LNDR	$C^l_1\left(\frac{2}{3} - \frac{2}{(2+a^2)}\right)$	$C^l_1\left(\frac{2}{3} - \frac{2a^2}{(2+a^2)}\right)$
FLT	$\left(C^f_1 A^{\frac{1}{2}} A_2 + 1\right) \times \left(\frac{2}{3} - \frac{2}{(2+a^2)}\right) + \frac{1}{6}C^f_1 C'_1 A^{\frac{1}{2}} A_2^2$	$\left(C^f_1 A^{\frac{1}{2}} A_2 + 1\right) \times \left(\frac{2}{3} - \frac{2a^2}{(2+a^2)}\right) - \frac{1}{3}C^f_1 C'_1 A^{\frac{1}{2}} A_2^2$
SSG	$C^s_1\left(\frac{2}{3} - \frac{2}{(2+a^2)}\right) - \frac{1}{6}C^s_2 A_2$	$C^s_1\left(\frac{2}{3} - \frac{2a^2}{(2+a^2)}\right) + \frac{1}{3}C^s_2 A_2$

where $A_2 = \dfrac{8(1-a^2)^2}{3(2+a^2)^2}$

Table 3 Components of the diffusion models valid for ZMS turbulence

Model	$\mathcal{D}^c_{11}$	$\mathcal{D}^c_{33}$
DH	$3C^{dh}_s a^2(2+a^2)\mathcal{K}$	$3C^{dh}_s a^4(2+a^2)\mathcal{K}$
HL	$3C^{hl}_s a^2(2+a^2)\mathcal{K}$	$9C^{hl}_s a^4(2+a^2)\mathcal{K}$
MH	$\frac{3}{2}C^{mh}_s(2+a^2)^2\mathcal{K}$	$\frac{9}{2}C^{mh}_s a^2(2+a^2)^2\mathcal{K}$
LUM	$3C^l_{s1}a^2(2+a^2)(1+2C^l_{s2} + 3a^2C^l_{s2})\mathcal{K}$	$3C^l_{s1}a^2(2+a^2)[(3a^2+6C^l_{s2} + 9a^2C^l_{s2}) - P_D(4+6a^2 + 20C_{s2} + 30a^2C_{s2})]\mathcal{K}$

where $\mathcal{K} = \beta B^3/\mathcal{A}x_3^4$

Table 4 Summary of anisotropy levels ($a = (\overline{u_3u_3}/\overline{u_1u_1})^{1/2}$) predicted by various combinations of pressure–strain and diffusion models

Model	DH	HL	MH	LUM
LNDR	1.00	1.75	1.75	1.00
FLT	1.00	1.54	1.54	1.00
SSG	1.00	6.88	6.88	1.00

The dissipation rate is approximated as[34]

$$\epsilon = \mathcal{A}\frac{\bar{u}_1^3}{l} = \frac{\mathcal{A}B^3}{\beta x_3^4} \tag{27}$$

These expressions are then substituted into the components for each of the pressure–strain and diffusion models to be evaluated. Table 2 contains a summary of the pressure–strain components, and Table 3 contains a summary of the diffusion components for ZMS turbulence. Note that all of the components in Tables 2 and 3 are functions of a only. The additional coefficient $\mathcal{K}$, which appears in every diffusion component, was formed to simplify the component expressions and because it cancels out in the next step of the analysis. In the LUM model, the terms resulting from the triple correlation model and the pressure diffusion model are given separately. The coefficient P_D, for which LUM assigns the value $P_D = \frac{1}{5}$, multiplies the terms resulting from the pressure–diffusion model.

Finally, each combination of pressure–strain and diffusion models is substituted into Eq. (22) to obtain an implicit expression for a. As an example, the combination of the LNDR pressure–strain model and the HL diffusion model gives

$$a^2 = \frac{\left[\frac{2}{3} - \left(\frac{2}{3} - [2a^2/(2+a^2)]\right)C^l_1\right]}{3\left[\frac{2}{3} - \left(\frac{2}{3} - [2/(2+a^2)]\right)C^l_1\right]} \tag{28}$$

which can be solved to predict an anisotropy $a = 1.75$. The summary of predicted anisotropies for all combinations of models is given in Table 4.

It is evident from Table 4 that large discrepancies exist between the predicted anisotropy levels of the different models and that no combination of models gives a result that is correct for ZMS turbulence

($a \approx 1.2$). The DH model predicts isotropic turbulence for all cases because there is no mechanism in the diffusion model to generate anisotropy. This result is clearly not plausible because anisotropy must exist for transport to occur. The HL and MH models happen to predict identical results for ZMS turbulence because of the factor-of-three difference between their respective diffusion components. Although these models both generate anisotropy, the factor of three is evidently much too large because no existing pressure–strain relation has enough restorative effect to reduce the anisotropy even close to the correct level. The LUM model does generate anisotropy, but the pressure–diffusion component eliminates the anisotropy generated in weakly inhomogeneous turbulence. Thus, for ZMS turbulence the LUM model predicts $a = 1.0$ regardless of the pressure–strain relation used, a result that, like that from the DH model, is not plausible. To verify the amount of anisotropy that is generated by the LUM model, when pressure–diffusion is neglected, i.e., when P_D is set to zero, the model predicts anisotropies of $a = 1.54$, 1.48, and 1.84 when combined with the LNDR, FLT, and SSG pressure–strain models, respectively. Thus, the LUM model generates less anisotropy than the HL and MH models, and like the HL and MH models, the resulting anisotropy level depends on the pressure–strain relation used.

The predictions obtained using the DH model and the MH model are not unexpected on the basis of previous evaluations; however, the predictions obtained using the HL and LUM models are surprisingly poor. The poor predictions of the HL and LUM models suggest that there may still be some fundamental error in these diffusion relations. Whereas this error could easily be attributed to the gradient-transport form of the model relations, a closer look at the structure of ZMS turbulence will show that this is not the case. In fact, there is no other flow where the assumption of gradient transport is more appropriate. In the next section, ZMS turbulence is examined in much greater detail by analyzing the moment equations of turbulence. This analysis provides insight into each of the different diffusion processes and, thus, enables us to make a recommendation of the best existing diffusion model.

B. Analysis of the Moment Equations in ZMS Turbulence

On the basis of the experimental data for ZMS turbulence, there is no question that anisotropy is generated in diffusive turbulence. The question is how much anisotropy is generated and what, if any, effect pressure–diffusion has on the resulting level of anisotropy. In this section, this question is addressed by examining the roles and the relative influence of $\overline{u_i u_j u_k}$ and pressure–diffusion in ZMS turbulence.

The constant level of anisotropy is the characteristic of ZMS turbulence around which much of the following analysis is focused. The balance that maintains the anisotropy level contains very few processes, permitting insight into the behavior of each process. The behavior and the relative influence of each process is established here through an examination of the second- and third-moment transport equations for turbulence.

The equations for $\overline{u_1 u_1}$, $\overline{u_2 u_2}$, and $\overline{u_3 u_3}$ in ZMS turbulence are obtained from a reduction of Eq. (1) and are, respectively,

$$0 = -\frac{\partial}{\partial x_3}(\overline{u_1 u_1 u_3}) - C_1 \frac{\epsilon}{k}\left(\overline{u_1 u_1} - \frac{2}{3}k\right) - \frac{2}{3}\epsilon \tag{29}$$

$$0 = -\frac{\partial}{\partial x_3}(\overline{u_2 u_2 u_3}) - C_1 \frac{\epsilon}{k}\left(\overline{u_2 u_2} - \frac{2}{3}k\right) - \frac{2}{3}\epsilon \tag{30}$$

$$0 = -\frac{\partial}{\partial x_3}\left(\overline{u_3 u_3 u_3} + 2\frac{\overline{p' u_3}}{\rho}\right) - C_1 \frac{\epsilon}{k}\left(\overline{u_3 u_3} - \frac{2}{3}k\right) - \frac{2}{3}\epsilon \tag{31}$$

where $\mathcal{D}_{ij}$ is given in its exact form but neglecting molecular transport. For simplicity, the Rotta[3] pressure–strain relation has been used in this analysis. It is evident from Eqs. (29–31) that the anisotropy level is maintained by a balance of turbulent transport, pressure–diffusion, and the return-to-isotropy (RTI) portion of pressure–strain. Whereas the dissipation rate is part of each RS balance, its contribution is the same in each equation and, hence, dissipation does not influence anisotropy. The influence of RTI in Eqs. (29–31) is to restore isotropy, and this is accomplished by means of a simple redistribution of turbulent kinetic energy among the normal stress (energy) components.

The influence of turbulent transport in Eqs. (29–31) is established by consideration of the exact transport equation for the triple correlation. For ZMS turbulence, the exact equation for the transport of $\overline{u_i u_j u_k}$ is

$$\begin{aligned}0 = &-\frac{\partial}{\partial x_l}\left(\overline{u_i u_j u_k u_l} + \overline{\frac{p'}{\rho}u_j u_k}\delta_{il} + \overline{\frac{p'}{\rho}u_i u_k}\delta_{jl} + \overline{\frac{p'}{\rho}u_i u_j}\delta_{kl}\right)\\ &+ \overline{u_i u_j}\frac{\partial \overline{u_k u_l}}{\partial x_l} + \overline{u_i u_k}\frac{\partial \overline{u_j u_l}}{\partial x_l} + \overline{u_j u_k}\frac{\partial \overline{u_i u_l}}{\partial x_l}\\ &+ \overline{\frac{p'}{\rho}\left(\frac{\partial u_i u_j}{\partial x_k} + \frac{\partial u_i u_k}{\partial x_j} + \frac{\partial u_j u_k}{\partial x_i}\right)}\end{aligned} \tag{32}$$

where the molecular diffusion and dissipation of $\overline{u_i u_j u_k}$ have been neglected on the basis of high-turbulence Reynolds number. The triple correlations are established by a balance of the quadruple correlation, which represents the transport of $\overline{u_i u_j u_k}$ by turbulent fluctuations, the velocity-weighted moment of pressure–diffusion, the production of $\overline{u_i u_j u_k}$ by turbulent fluctuations, and the modification of $\overline{u_i u_j u_k}$ due to pressure interactions. Following the work of Hanjalić and Launder[4] and all subsequent studies, the quadruple correlation is simplified using the quasinormal approximation, and the modification due to pressure is approximated using a third-moment RTI relation. Equation (32) is then reduced to

$$\begin{aligned}0 = &\,\overline{u_i u_l}\frac{\partial \overline{u_j u_k}}{\partial x_l} + \overline{u_j u_l}\frac{\partial \overline{u_i u_k}}{\partial x_l} + \overline{u_k u_l}\frac{\partial \overline{u_i u_j}}{\partial x_l}\\ &- \frac{\partial}{\partial x_l}\left(\overline{\frac{p'}{\rho}u_j u_k}\delta_{il} + \overline{\frac{p'}{\rho}u_i u_k}\delta_{jl} + \overline{\frac{p'}{\rho}u_i u_j}\delta_{kl}\right)\\ &+ \frac{1}{C_{\pi_1}}\frac{\epsilon}{k}\overline{u_i u_j u_k}\end{aligned} \tag{33}$$

The component equations for the triple correlations appearing in Eqs. (29–31), $\overline{u_1 u_1 u_3}$, $\overline{u_2 u_2 u_3}$, and $\overline{u_3 u_3 u_3}$, respectively, are obtained from Eq. (33) and are

$$0 = -\overline{u_3 u_3}\frac{\partial \overline{u_1 u_1}}{\partial x_3} - \frac{\partial}{\partial x_3}\left(\overline{\frac{p'}{\rho}u_1 u_1}\right) + \frac{1}{C_{\pi_1}}\frac{\epsilon}{k}\overline{u_1 u_1 u_3} \tag{34}$$

$$0 = -\overline{u_3 u_3}\frac{\partial \overline{u_2 u_2}}{\partial x_3} - \frac{\partial}{\partial x_3}\left(\overline{\frac{p'}{\rho}u_2 u_2}\right) + \frac{1}{C_{\pi_1}}\frac{\epsilon}{k}\overline{u_2 u_2 u_3} \tag{35}$$

$$0 = -3\overline{u_3 u_3}\frac{\partial \overline{u_3 u_3}}{\partial x_3} - \frac{\partial}{\partial x_3}\left(3\overline{\frac{p'}{\rho}u_3 u_3}\right) + \frac{1}{C_{\pi_1}}\frac{\epsilon}{k}\overline{u_3 u_3 u_3} \tag{36}$$

It is evident from Eqs. (34–36) that the diagonal triple correlation $\overline{u_3 u_3 u_3}$ has a magnitude different from that of the $\overline{u_1 u_1 u_3}$ and $\overline{u_2 u_2 u_3}$ components (which are equal) and, thus, turbulent transport, in Eqs. (29–31), has the effect of generating anisotropy (as suggested in the preceding sections of this paper).

The only remaining term that appears to affect the RS balances given in Eqs. (29–31) is the pressure–diffusion term in Eq. (31). It is expected that this term has some influence on the turbulence structure, particularly in diffusive turbulence. In the LUM model, the pressure–diffusion component is 20% of the magnitude of its related turbulent transport component. We speculate here that it is not only pressure–diffusion that is important but also the velocity weighted moments of pressure–diffusion that appear in Eqs. (34–36). The influence of both of these pressure–velocity processes is deduced next.

Before considering the influence of the individual pressure–velocity processes, it is very straightforward to demonstrate the importance of the pressure–velocity terms in the diffusive limit by simply applying the conventional hypothesis that all pressure–velocity terms are negligible. By applying this hypothesis, Eq. (33) becomes

$$-\overline{u_i u_j u_k} = C_{\pi_1}\frac{k}{\epsilon}\left(\overline{u_i u_l}\frac{\partial \overline{u_j u_k}}{\partial x_l} + \overline{u_j u_l}\frac{\partial \overline{u_i u_k}}{\partial x_l} + \overline{u_k u_l}\frac{\partial \overline{u_i u_j}}{\partial x_l}\right) \tag{37}$$

which is exactly the HL model (if $C_{\pi_1} = 0.11$). We know from the evaluations given in the preceding section that the HL model gives extremely poor predictions of the anisotropy in ZMS turbulence regardless of the pressure–strain relation used. Thus, the hypothesis that pressure–velocity terms are negligible must be rejected.

Next, consider the pressure–diffusion process itself and the RS balances given by Eqs. (29–31). We can show that pressure–diffusion results in a net gain or loss of k by summing Eqs. (29–31) (and dividing by 2) to give

$$0 = -\frac{\partial}{\partial x_3}\left(\overline{ku_3} + \frac{\overline{p'u_3}}{\rho}\right) - \epsilon \tag{38}$$

A limited amount of experimental evidence for jets[35] suggests that the pressure–velocity correlation is negative. This, combined with the magnitude of $\overline{p'u_3}$ decreasing in the direction of transport, suggests that pressure–diffusion results in a loss of k and, thus, reduces anisotropy.

The influence of the velocity-weighted moment of pressure–diffusion is slightly more difficult to ascertain because nothing is presently known about the triple correlations involving pressure fluctuations. We can, however, deduce the influence of this process on the basis of the preceding analysis. Consider, first, that the HL model emerges naturally from the $\overline{u_i u_j u_k}$ transport equation (in ZMS turbulence) if the pressure–velocity processes are neglected and that the HL model gives a factor-of-three difference between $\overline{u_3u_3u_3}$ and $\overline{u_1u_1u_3}$, which results in an anisotropy of $a = 1.75$. To reduce such a high level of anisotropy by pressure–diffusion alone, the pressure–velocity correlation in Eq. (31) would need to have a magnitude nearly 60% as high as the triple correlation $\overline{u_3u_3u_3}$; a value that is unrealistically high. Thus, the velocity-weighted moments of pressure–diffusion, which appear in all triple correlation equations, must be a mechanism that regulates the generation of anisotropy at the third-moment level.

To summarize, pressure–diffusion and the velocity-weighted moments of pressure–diffusion have the combined effect of reducing the anisotropy in diffusive turbulence. The velocity-weighted moment of pressure–diffusion regulates the generation of anisotropy, and pressure–diffusion itself has the effect of attenuating the anisotropy that is generated. Clearly, a diffusion model must contain mechanisms that can account for these pressure–velocity effects to make realistic flow predictions in regions where the diffusive limit is approached. In the final section of this paper, a recommendation of the best existing diffusion is made on the basis of the analysis given next.

V. Discussion

The analysis given in the preceding section offered some concrete evidence that the pressure–velocity processes in the second- and third-moment equations are crucial in the RS and $\overline{u_i u_j u_k}$ balances as the diffusive limit is approached. Furthermore, from the evaluation of existing $\mathcal{D}_{ij}$ models and the analysis given in the preceding section, it is evident that the LUM diffusion model is the only existing model that has a form capable of accounting for the pressure–velocity processes in a physically realistic manner. In words, the LUM triple correlation model has a component that links each of the triple correlations and, thus, controls the generation of anisotropy, and the LUM pressure–diffusion model attenuates the anisotropy that is generated. In its present form, however, the LUM model eliminates the anisotropy generated in weakly inhomogeneous turbulence. On the basis of the findings presented in this paper, we suggest a modification to the LUM model that enables exact predictions for the diffusive limit.

The modification is simply to reduce the influence of the pressure–diffusion term, thereby allowing anisotropy to exist in the diffusive limit. If the coefficient P_D in the pressure–diffusion model is reduced from $\frac{1}{5}$ to $\frac{4}{25}$ (a reduction by $\frac{4}{5}$), the anisotropy in the diffusive limit is predicted to be $a = 1.2$, the level suggested by Hopfinger and Toly.[26] For completeness, Table 4 is repeated in Table 5 to show the results of the modified Lumley model combined with each of the three pressure–strain relations and to contrast these results with the results of the other diffusion models. The modified LUM model gives reasonable results when combined with each of the LNDR, FLT, and SSG models, demonstrating that the modified model is still compatible with all of the popular pressure–strain relations. In fact, the range of anisotropy predictions for ZMS turbulence is less than 10% across the spectrum of π_{ij} models tested. This suggests that, for diffusive turbulence, anisotropy may be controlled more by the diffusion process than by pressure–strain.

Table 5 Summary of anisotropy levels [$a = (\overline{u_3u_3}/\overline{u_1u_1})^{1/2}$] predicted by various combinations of pressure–strain and diffusion models

Model	DH	MH	HL	LUM original	LUM modified
LNDR	1.00	1.75	1.75	1.00	1.20
FLT	1.00	1.54	1.54	1.00	1.30
SSG	1.00	6.88	6.88	1.00	1.25

The modification to the LUM model is not yet intended to be viewed as a strict rule but rather as a suggestion that enables the model to predict the correct limiting behavior. A more rigorous approach is to consider the LUM model as a gradient transport expression that conforms to all of the rules of tensor mathematics and that contains components that account for each important process identified in our present analysis. The three coefficients C_{s1}^l, C_{s2}^l, and P_D could be considered adjustable and then tuned to give the correct physical influence for each process. (Note that even though there were no unknown coefficients generated in its original derivation, several approximations were necessary to establish the final form of the LUM model. Thus, the model is only absolutely nonadjustable for the case where all of the approximations are exact.) Modifications of this magnitude, however, would be justified only if more detailed experimental data were available for diffusive turbulence.

Finally, our conclusion that the LUM diffusion model is the best existing gradient-transport relation for diffusion is not the first. It is, however, the most compelling because our conclusion is based on a purely diffusive flow. Recall that, on the basis of flow predictions, Schwarz and Bradshaw[20] also concluded that the LUM model was the best existing diffusion model. Nevertheless, use of the LUM model in computations using RS turbulence closures has not been widespread. This lack of use is undoubtedly a result of the supposed complexity of implementation of this model as compared to the more popular, but less realistic, DH model. Note that all of the models considered in the present evaluation of ZMS turbulence (DH, HL, MH, and LUM) were classified as simple models on the basis that they were derived using the gradient-transport hypothesis and that their implementation was considered relatively simple compared with those models classified as intermediate and complex. The only difference, in terms of implementation, between the DH, MH, HL, and LUM models is the number of terms to be resolved.

VI. Conclusions

An investigation of diffusion modeling in second-moment turbulence closures has been presented. The main focus was to establish the relative importance of the different processes in $\mathcal{D}_{ij}$ and then to assess existing diffusion models in terms of their ability to model the entire diffusion process. ZMS turbulence was used as a benchmark for the evaluation of existing models, for establishing the influence of the pressure–velocity processes in $\mathcal{D}_{ij}$, and for the final assessment of existing diffusion models. The main conclusions follow.

1) Existing diffusion models based solely on the triple velocity correlation give poor predictions in the diffusive limit. The main weakness of these models was shown to be the neglect or inadequate handling of the pressure–velocity processes in $\mathcal{D}_{ij}$ and $\overline{u_i u_j u_k}$.

2) Pressure–diffusion and the velocity-weighted moment of pressure–diffusion play a central role in establishing anisotropy in diffusive turbulence. The role of the velocity-weighted moment of pressure–diffusion is to regulate the generation of anisotropy, and the role of pressure–diffusion is to reduce the anisotropy that is generated. Although this was not made evident using the flow data from previous model evaluations, the present evaluation and analysis of ZMS turbulence made the role of the pressure–velocity processes in $\mathcal{D}_{ij}$ clear.

3) The present evaluation of diffusion models using ZMS turbulence and the subsequent physical assessment of models suggest that the best existing relation for modeling diffusive transport in

the $\overline{u_i u_j}$ equation is the LUM model. The LUM model has separate mechanisms to account for the triple correlation, pressure–diffusion, and the velocity-weighted moment of pressure–diffusion. A modification suggested in this paper was to adjust the coefficient in the pressure–diffusion model to allow anisotropy to exist in the diffusive limit. The justification for this modification was taken from the present analysis of ZMS turbulence.

Appendix: Discussion on ZMS Turbulence

A brief discussion of the inverse power law for $\bar{u}_1$ and the turbulent Reynolds number in ZMS turbulence is given.

Using the expressions for $\bar{u}_1$ and l given in Eq. (17) and the experimental value for n, a Reynolds number based on the integral length scale can be formed as

$$Re_l = \bar{u}_1 l/\nu = \text{const} \tag{A1}$$

which suggests that the turbulence does not die out, even at distances far away from the source. Because this is not physically reasonable, it is more likely that n is slightly greater than 1.0 or that there are two decay laws for $\bar{u}_1$: one for the region close to the source, where the high-turbulence Reynolds number exists, and one for the region far from the source, where low-turbulence-Reynolds-number effects are evident. Given the experimental support for $n = 1.0$, it is reasonable here to assume that the inverse power law for $\bar{u}_1$ is valid for a considerable distance from the source.

Verification of the high-turbulence-Reynolds-number assumption is established by considering the benchmark measurements reported by Hopfinger and Toly[26] for the Taylor microscales λ in ZMS turbulence. Using these measurements, the Kolmogoroff or dissipating scales l_k are calculated by forming a Reynolds number based on the Taylor microscale ($Re_\lambda = \bar{u}_1\lambda/\nu$), estimating the dissipation rate from $\epsilon = 10 \times k^{3/2}/Re_\lambda \times \lambda$ [or $\epsilon = \mathcal{A}\bar{u}_1^3/l$ and then using $l_k = (\nu^3/\epsilon)^{1/4}$]. Calculation of these scales verifies that almost two orders of magnitude exist between λ and l_k and, thus, the high-turbulence-Reynolds-number approximation is justified in the region where the inverse power law for $\bar{u}_1$ is valid.

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